LINK BETWEEN TWO FACTORIZATIONS OF THE ZETA FUNCTIONS OF DWORK HYPERSURFACES

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Abstract. The aim of this article is to relate two different factorizations of the zeta functions of Dwork hypersurfaces which were obtained in two previous articles. The first factorization is explicit, given in terms of numerators of zeta functions of hypersurfaces of hypergeometric type. The second comes from an isotypic decomposition of the cohomology. To relate these two factorizations, we use a technique based on L functions of representations, following a method of Katz.

1. Introduction

Let $\mathbb{F}_q$ be a finite field of characteristic p having q elements and n a prime number ≥ 5 such that $q \equiv 1 \mod n$. In [8], Wan showed that the zeta function of the projective hypersurface $X_\psi \subset \mathbb{P}^{n-1}$ defined by $x_1^n + \cdots + x_n^n - n\psi x_1 \ldots x_n = 0$ (where $\psi \in \mathbb{F}_q^*$ is a parameter satisfying $\psi^n \neq 1$ so that X_ψ is non-singular) is

$$Z_{X_\psi/\mathbb{F}_q}(t) = \frac{(Q(t,\psi)R(qt,\psi))^{(-1)^{n-1}}}{(1-t)(1-qt)\ldots(1-q^{n-2}t)},$$

where Q is a polynomial of degree $n-1$ with integer coefficients which comes from mirror symmetry (more precisely, this factor appears in the zeta function of the quotient X_ψ/A where A is the group of roots of unity acting on X_ψ and defined below; see [8] for more details), and R is a polynomial of degree $\frac{1}{n}[(n-1)^n + (-1)^n(n-1)] - (n-1)$ with integer coefficients and with roots of absolute value $q^{-(n-4)/2}$.

In [3] and [4], we obtained two different factorizations of the polynomial R. The aim of this article is to compare them. More precisely, define

$$A = \{(\zeta_1, \ldots, \zeta_n) \in \mathbb{F}_q^n \mid \zeta_i^n = 1,\ \zeta_1 \ldots \zeta_n = 1\}/\{(\zeta, \ldots, \zeta)\};$$
$$\hat{A} = \{(a_1, \ldots, a_n) \in (\mathbb{Z}/n\mathbb{Z})^n \mid a_1 + \cdots + a_n = 0\}/\{(a, \ldots, a)\}.$$

The group A acts on X_ψ by coordinate-wise multiplication. Fix a prime $\ell \neq p$; as $q \equiv 1 \mod n$, $\boldsymbol{\mu}_n(\mathbb{F}_q) \simeq \boldsymbol{\mu}_n(\overline{\mathbb{Q}}_\ell)$ (where $\boldsymbol{\mu}_n(\mathbb{K})$ denotes the group of n-th roots of unity of $\mathbb{K}$) and denote by θ such an isomorphism. We identify $\hat{A}$ to the group of characters of A taking values in $\overline{\mathbb{Q}}_\ell$ thanks to the isomorphism $[a_1, \ldots, a_n] \mapsto ([\zeta_1, \ldots, \zeta_n] \mapsto \theta(\zeta_1)^{a_1} \ldots \theta(\zeta_n)^{a_n})$. With this identification, we write $a([\zeta]) = a_1(\zeta_1) \ldots a_n(\zeta_n)$ where $a_i(\zeta_i) = \theta(\zeta_i)^{a_i}$. We also set $\hat{A}^* = \hat{A} \setminus \{[0]\}$. Given $a \in \hat{A}$, we introduce the following notations from [3, 4]:

- $m_a = |\mathbb{Z}/n\mathbb{Z} \setminus \{a_1, \ldots, a_n\}|$; it's (see [4, §3.3]) the multiplicity of the character a appearing in the $\overline{\mathbb{Q}}_\ell[A]$-module $H^{n-2}_{\text{et}}(\overline{X}_\psi, \overline{\mathbb{Q}}_\ell)$;
- γ_a = number of permutations of $(a_1, \ldots, a_n)$;

- $S_{\overline{a}} = \{\sigma \in \mathfrak{S}_n \mid \exists k \in (\mathbb{Z}/n\mathbb{Z})^\times, {}^\sigma a = a\}$; because n is prime, if $\sigma \in S_{\overline{a}}$ and $a \neq [0]$, there exist a unique $k \in (\mathbb{Z}/n\mathbb{Z})^\times$ such that ${}^\sigma a = ka$; the reason for the bar over the a is to use the same notation as in [3, §5.1];
- k_a is the application $S_{\overline{a}} \to (\mathbb{Z}/n\mathbb{Z})^\times$, $\sigma \mapsto k$ thus defined when $a \neq [0]$.

In [3], we showed that the polynomial R can be factored as[1]

$$R(t) = \prod_{a \in (\mathbb{Z}/n\mathbb{Z})^\times \times \mathfrak{S}_n \backslash \hat{A}^*} R_a(t)^{\gamma_a / |\mathrm{Im}\, k_a|},$$

where R_a are polynomials which appear (up to a multiplicative factor affecting their variable) in the numerator of the zeta function of a hypersurface of hypergeometric type of which we can give an explicit equation (see [3, §5.3 and §3.2]). As these hypergeometric hypersurfaces are not smooth, the degree of the factors R_a are not automatically known; as a consequence of the main result of this article, we will obtain $\deg R_a$ (see Corollary 5.5).

In [4], we showed that the polynomial R can be factored as[2]

$$R(t) = \prod_{a \in (\mathbb{Z}/n\mathbb{Z})^\times \times \mathfrak{S}_n \backslash \hat{A}^*} Q_a(t)^{\gamma_a},$$

where the polynomials $Q_a = Q_{a,1}$ have degree $m_a(n-1)/|\mathrm{Im}\, k_a|$ and satisfy

$$Q_a(t)^{\gamma_a} = \det(1 - t\,\mathrm{Frob}^* | H^{n-2}_{\mathrm{et}}(\overline{X}_\psi, \mathbb{Q}_\ell)^{W_a}),$$

where $W_a = W_{a,1}$ is an irreducible representation over $\mathbb{Q}$ of the automorphism group $A \rtimes \mathfrak{S}_n$ of X_ψ and $H^{n-2}_{\mathrm{et}}(\overline{X}_\psi, \mathbb{Q}_\ell)^{W_a}$ is the isotypic component of type W_a of the $\mathbb{Q}[A \rtimes \mathfrak{S}_n]$-module $H^{n-2}_{\mathrm{et}}(\overline{X}_\psi, \mathbb{Q}_\ell)$.

We now describe our method to relate these two factorizations. It is the same as Katz used for Artin-Schreier curves in [6] (it is also used by Wan in [8, Lemma 7.2] to show the existence of the polynomials Q and R). First, we compute, for $a \in \hat{A}$, the following sums, which belong to $\overline{\mathbb{Q}}_\ell$,

$$S_{X_\psi/\mathbb{F}_q,a,r} = \frac{1}{|A|} \sum_{[\zeta] \in A} a([\zeta])\, |\mathrm{Fix}(\mathrm{Frob}^r \circ [\zeta]^{-1})|.$$

(Here Frob denotes the Frobenius induced by $x \mapsto x^q$ and $\mathrm{Fix}(f)$ denotes the set of elements of $\overline{X}_\psi$ fixed by the endomorphism f of $\overline{X}_\psi$.) Next, we consider the corresponding L function

$$L_{X/\mathbb{F}_q,a}(t) = \exp\left(\sum_{r=1}^{+\infty} S(X/\mathbb{F}_q, a, r)\, \frac{t^r}{r}\right),$$

The computation of the sums $S_{X_\psi/\mathbb{F}_q,a,r}$ (see Theorem 3.4) will allow to relate $R_a(t)$ and $L_{X/\mathbb{F}_q,a}(t)$. Moreover, a trace formula and the fact that A acts trivially on the spaces $H^i_{\mathrm{et}}(X_\psi, \mathbb{Q}_\ell)$ for $i \neq n-2$ (see §4) will show that, when $a \neq [0]$,

$$L_{X/\mathbb{F}_q,a}(t) = \det(1 - t\,\mathrm{Frob}^* | H^{n-2}_{\mathrm{et}}(\overline{X}_\psi, \overline{\mathbb{Q}}_\ell)^a),$$

where $H^{n-2}_{\mathrm{et}}(\overline{X}_\psi, \overline{\mathbb{Q}}_\ell)^a$ is the isotypic component of type a of $H^{n-2}_{\mathrm{et}}(\overline{X}_\psi, \overline{\mathbb{Q}}_\ell)$. This will allow us to relate Q_a to $L_{X/\mathbb{F}_q,a}(t)$ and thus to $R_a(t)$. The final result is that

$$R_a(t) = Q_a(t)^{|\mathrm{Im}\, k_a|}.$$

[1] With the notation of this article, we have $|\mathrm{Im}\, k_a| = K_a$ when $a \neq [0]$ and n is prime.

[2] Because n is prime, the formulas simplify greatly.

The article is organized as follows. To compute the sums $S_{X_\psi/\mathbb{F}_q,a,r}$ for the Dwork hypersurfaces in §3, we first need to determine them for Fermat hypersurfaces (see §2). After recalling the properties of L functions in §4, we establish the link between R_a and Q_a in §5. The results and notations from [3] and [4] are only used in §5.

2. Computation of the sums for Fermat hypersurfaces

Let us first note that, as $S_{X/\mathbb{F}_q,a,r} = S_{X/\mathbb{F}_{q^r},a,1}$, we only need to deal with the case $r = 1$, i.e. compute $S_{X/\mathbb{F}_q,a} = S_{X/\mathbb{F}_q,a,1}$. We will restrict ourself to this case in all of this section.

Let $d \geq 1$ be an integer such that $q \equiv 1 \mod d$. We consider the hypersurface $D \subset \mathbb{P}^{n-1}$ defined by $x_1^d + \ldots + x_n^d = 0$ and denote by D^* the corresponding toric hypersurface (i.e. with all coordinates non zero).

We adapt the notations of the introduction to Fermat hypersurfaces (when $d = n$, they correspond to the case $\psi = 0$ of Dwork hypersurfaces) by setting

$$\begin{aligned} A &= \{(\zeta_1, \ldots, \zeta_n) \in \mathbb{F}_q^n \mid \zeta_i^d = 1,\ \zeta_1 \ldots \zeta_n = 1\}/\{(\zeta, \ldots, \zeta)\};\\ \hat{A} &= \{(a_1, \ldots, a_n) \in (\mathbb{Z}/d\mathbb{Z})^n \mid a_1 + \cdots + a_n = 0\}/\{(a, \ldots, a)\}, \end{aligned}$$

and identifying $\hat{A}$ to the group of characters of A taking values in $\overline{\mathbb{Q}}_\ell$ thanks to a fixed isomorphism between $\boldsymbol{\mu}_d(\mathbb{F}_q)$ and $\boldsymbol{\mu}_d(\overline{\mathbb{Q}}_\ell)$.

The map from $\mathrm{Hom}(\boldsymbol{\mu}_d(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ to $\{\chi \in \widehat{\mathbb{F}_q^*} \mid \chi^d = \mathbf{1}\}$ which takes b to $\check{b}\colon x \mapsto b(x^{(q-1)/d})$ is a group isomorphism; we denote its inverse by $\chi \mapsto \hat{\chi}$.

Finally, Frob denotes the endomorphism of D induced by $x \mapsto x^q$, and, if $a \in \hat{A}$, we consider

$$S_{D/\mathbb{F}_q,a} = \frac{1}{|A|} \sum_{[\zeta] \in A} a([\zeta])\, |\mathrm{Fix}_D(\mathrm{Frob} \circ [\zeta]^{-1})|;$$

$$S_{D^*/\mathbb{F}_q,a} = \frac{1}{|A|} \sum_{[\zeta] \in A} a([\zeta])\, |\mathrm{Fix}_{D^*}(\mathrm{Frob} \circ [\zeta]^{-1})|.$$

The method we are going to use to compute $S_{D/\mathbb{F}_q,a}$ and $S_{D^*/\mathbb{F}_q,a}$ is an adaptation of the one used by Katz for Artin-Schreier curves in [6]; it amounts to adapting the classical formula for the number of points over $\mathbb{F}_q$ of D and D^* (see for example [1]).

2.1. Preliminary results.

We begin with remarks which we will use constantly in what follows.

Remark 2.1.

(1) If $x^{q-1} = \xi$ with $\xi^d = 1$, then $x^d \in \mathbb{F}_q$. Indeed, $(x^d)^{q-1} = (x^{q-1})^d = \xi^d = 1$.
(2) If $\xi^d = 1$, then every $y \in \overline{\mathbb{F}}_q$ satisfying $y^{(q-1)/d} = \xi$ belongs to $\mathbb{F}_q$. Indeed, $y^{q-1} = (y^{(q-1)/d})^d = \xi^d = 1$.
(3) If $\xi^d = 1$, $\chi^d = \mathbf{1}$ and $y^{(q-1)/d} = \xi$, then $\chi(y)$ is independent of the choice of y. Indeed, with the preceding notations, $\chi(y) = \hat{\chi}(y^{(q-1)/d}) = \hat{\chi}(\xi)$.

Lemma 2.2. *If $\xi \in \mathbb{F}_q$ satisfies $\xi^d = 1$, then, using the previous notations,*

$$\forall \eta \in \widehat{\mathbb{F}_q^*}, \qquad \sum_{x^{q-1} = \xi} \eta(x^d) = \begin{cases} (q-1)\hat{\eta}(\xi) & \text{if } \eta^d = \mathbf{1}, \\ 0 & \text{if } \eta^d \neq \mathbf{1}. \end{cases}$$

Proof. Let $y \in \mathbb{F}_q$ be such that $y^{(q-1)/d} = \xi$. We extend η into a character $\overline{\eta}$ of $\overline{\mathbb{F}}_q^*$ and choose $\xi' \in \overline{\mathbb{F}}_q$ such that $\xi'^d = y$. By making the change of variable $x = \xi' z$, we obtain

$$\sum_{x^{q-1}=\xi} \eta(x^d) = \overline{\eta}(\xi'^d) \sum_{z^{q-1}=1} \eta^d(z) = \eta(y) \times \begin{cases} q-1 & \text{if } \eta^d = \mathbf{1}, \\ 0 & \text{if } \eta^d \neq \mathbf{1}, \end{cases}$$

with $\eta(y) = \hat{\eta}(\xi)$ by Remark 2.1.(3). $\square$

2.2. **Computation of the sums for Fermat hypersurfaces.** Before computing $S_{D/\mathbb{F}_q,a}$, we show a formula for the corresponding fixator.

Proposition 2.3. *Let φ be a fixed non-trivial additive character of $\mathbb{F}_q$. If $[\zeta] = [\zeta_1, \ldots, \zeta_n] \in A$, then, with the notations of the begining of §2,*

$$|\mathrm{Fix}_D(\mathrm{Frob} \circ [\zeta]^{-1})| = 1 + q + \cdots + q^{n-2} + \frac{1}{q} \sum_{\substack{\chi_i^d = \mathbf{1},\ \chi_i \neq \mathbf{1} \\ \chi_1 \cdots \chi_n = \mathbf{1}}} G(\varphi, \chi_1^{-1}) \ldots G(\varphi, \chi_n^{-1}) \hat{\chi}_1(\zeta_1) \ldots \hat{\chi}_n(\zeta_n).$$

Proof. We first compute the affine fixator, then deduce the projective one thanks to the formula

$$|\mathrm{Fix}_D^{\mathrm{proj}}(\mathrm{Frob} \circ [\zeta]^{-1})| = \frac{|\mathrm{Fix}_D^{\mathrm{aff}}(\mathrm{Frob} \circ \zeta^{-1})| - 1}{q-1}.$$

(Let us justify this quickly: if $[x_1^q : \ldots : x_n^q] = [\zeta_1 x_1 : \ldots : \zeta_n x_n]$ with one of the x_i non-zero, then $(x_1^q, \ldots, x_n^q) = \lambda(\zeta_1 x_1, \ldots, \zeta_n x_n)$ where $\lambda \in \overline{\mathbb{F}}_q^*$; thus, for i such that $x_i \neq 0$, $x_i^q = \lambda \zeta_i x_i$ and so, if $\mu \in \overline{\mathbb{F}}_q^*$, we have $(\mu x_i)^q = \lambda \zeta_i (\mu x_i) \iff \mu^{q-1} = \lambda$, equation which has $q-1$ solutions in $\overline{\mathbb{F}}_q$.)

Let $f(x) = x_1^d + \cdots + x_n^d$ so that D is the hypersurface defined by $f = 0$. As we have said, we take inspiration on the classical computation of $|D(\mathbb{F}_q)|$ as presented in [1]. Consider $(x_1, \ldots, x_n) \in \overline{\mathbb{F}}_q^n$ satisfying $x_i^q = \zeta_i x_i$. This means that either $x_i = 0$ or $x_i^{q-1} = \zeta_i$, thus $x_i^d \in \mathbb{F}_q$ in all cases by Remark 2.1.(1); in particular, $f(x) \in \mathbb{F}_q$. Using an orthogonality formula, we deduce that

$$|\mathrm{Fix}_D^{\mathrm{aff}}(\mathrm{Frob} \circ \zeta^{-1})| = \frac{1}{q} \sum_{a \in \mathbb{F}_q} \sum_{x_i^q = \zeta_i x_i} \varphi(a f(x)).$$

The first step, in order to make a Fourier inversion, is to obtain sums over non-zero elements:

$$\begin{aligned} &|\mathrm{Fix}_D^{\mathrm{aff}}(\mathrm{Frob} \circ \zeta^{-1})| \\ &\quad = q^{n-1} + \frac{1}{q} \sum_{a \in \mathbb{F}_q^*} \sum_{x_i^q = \zeta_i x_i} \varphi(a x_1^d) \ldots \varphi(a x_n^d) \\ &\quad = q^{n-1} + \frac{1}{q} \sum_{a \in \mathbb{F}_q^*} \left(1 + \sum_{x_1^{q-1} = \zeta_1} \varphi(a x_1^d)\right) \ldots \left(1 + \sum_{x_n^{q-1} = \zeta_n} \varphi(a x_n^d)\right). \end{aligned}$$

As all the ax_i^d are non-zero, we can use the Fourier inversion formula for the functions $\varphi|_{\mathbb{F}_q^*}$:

$$\begin{aligned}
&|\mathrm{Fix}_D^{\mathrm{aff}}(\mathrm{Frob}\circ\zeta^{-1})|\\
&\quad = q^{n-1} + \frac{1}{q}\sum_{a\in\mathbb{F}_q^*}\left(1+\frac{1}{q-1}\sum_{\eta_1\in\widehat{\mathbb{F}_q^*}}\sum_{x_1^{q-1}=\zeta_1} G(\varphi,\eta_1^{-1})\eta_1(ax_1^d)\right)\\
&\qquad\qquad \dots\left(1+\frac{1}{q-1}\sum_{\eta_n\in\widehat{\mathbb{F}_q^*}}\sum_{x_n^{q-1}=\zeta_n} G(\varphi,\eta_n^{-1})\eta_n(ax_n^d)\right).
\end{aligned}$$

As $G(\varphi,\mathbf{1})=-1$, this is equal to

$$\begin{aligned}
q^{n-1} + \frac{1}{q}\frac{1}{(q-1)^n}\sum_{\forall i,\ \eta_i\neq\mathbf{1}} G(\varphi,\eta_1^{-1})\dots G(\varphi,\eta_n^{-1})\left(\sum_{a\in\mathbb{F}_q^*}(\eta_1\dots\eta_n)(a)\right)&\\
\times\left(\sum_{x_1^{q-1}=\zeta_1}\eta_1(x_1^d)\right)\dots\left(\sum_{x_n^{q-1}=\zeta_n}\eta_n(x_n^d)\right)&.
\end{aligned}$$

The sum over a is immediate to compute thanks to an orthogonality formula and the sums over the x_i can be computed thanks to Lemma 2.2:

$$\sum_{a\in\mathbb{F}_q^*}(\eta_1\dots\eta_n)(a) = \begin{cases} q-1 & \text{if } \eta_1\dots\eta_n=\mathbf{1},\\ 0 & \text{if } \eta_1\dots\eta_n\neq\mathbf{1};\end{cases}$$

$$\sum_{x_i^{q-1}=\zeta_i}\eta_i(x_i^d) = \begin{cases} (q-1)\hat{\eta}_i(\zeta_i) & \text{if } \eta_i^d=\mathbf{1},\\ 0 & \text{if } \eta_i^d\neq\mathbf{1}.\end{cases}$$

Therefore,

$$\begin{aligned}
|\mathrm{Fix}_D^{\mathrm{aff}}(\mathrm{Frob}\circ\zeta^{-1})| &= q^{n-1}\\
&\quad + \frac{(q-1)}{q}\sum_{\substack{\chi_i^d=\mathbf{1},\ \chi_i\neq\mathbf{1}\\ \chi_1\cdots\chi_n=\mathbf{1}}} G(\varphi,\chi_1^{-1})\dots G(\varphi,\chi_n^{-1})\hat{\chi}_1(\zeta_1)\dots\hat{\chi}_n(\zeta_n).
\end{aligned}$$

Thus, in terms of projective fixator:

$$\begin{aligned}
|\mathrm{Fix}_D^{\mathrm{proj}}(\mathrm{Frob}\circ[\zeta]^{-1})| &= 1+q+\dots+q^{n-2}\\
&\quad + \frac{1}{q}\sum_{\substack{\chi_i^d=\mathbf{1},\ \chi_i\neq\mathbf{1}\\ \chi_1\cdots\chi_n=\mathbf{1}}} G(\varphi,\chi_1^{-1})\dots G(\varphi,\chi_n^{-1})\hat{\chi}_1(\zeta_1)\dots\hat{\chi}_n(\zeta_n). \quad\square
\end{aligned}$$

Before we give the next theorem, let us introduce a notation which we will often in what follows.

Notations. $\delta_P\in\{0,1\}$ is equal to 1 if and only if the property P is true; for example, $\delta_{a=[0]}$ equals 1 if $a=[0]$ and equals 0 if $a\neq[0]$.

Theorem 2.4. *We continue to consider a fixed non-trivial additive character φ of $\mathbb{F}_q$ and keep the notations of the beginning of §2. If $a \in \hat{A}$,*

$$S_{D/\mathbb{F}_q,a} = (1 + q + \cdots + q^{n-2})\delta_{a=[0]} + \frac{1}{q} \sum_{\chi^d=\mathbf{1},\ \chi\neq\check{a}_i} G(\varphi, \chi^{-1}\check{a}_1) \ldots G(\varphi, \chi^{-1}\check{a}_n).$$

Proof. By the definition of $S_{D/\mathbb{F}_q,a}$ and Proposition 2.3, we need to compute

$$\frac{1}{|A|} \sum_{[\zeta]\in A} a([\zeta])(1 + q + \cdots + q^{n-2}) + \frac{1}{q} \sum_{\substack{\chi_i^d=\mathbf{1},\ \chi_i\neq\mathbf{1} \\ \chi_1\cdots\chi_n=\mathbf{1}}} G(\varphi, \chi_1^{-1}) \ldots G(\varphi, \chi_n^{-1}) \frac{1}{|A|} \sum_{[\zeta]\in A} a([\zeta])\hat{\chi}_1(\zeta_1) \ldots \hat{\chi}_n(\zeta_n).$$

The value of the first sum results from an orthogonality formula:

$$\frac{1}{|A|} \sum_{[\zeta]\in A} a([\zeta]) = \delta_{a=[0]}.$$

The value of the second sum also results from orthogonality formulas:

$$\frac{1}{|A|} \sum_{[\zeta]\in A} a([\zeta])\hat{\chi}_1(\zeta_1) \ldots \hat{\chi}_n(\zeta_n) = \frac{1}{|A|} \sum_{[\zeta]\in A} (a_1\hat{\chi}_1)(\zeta_1) \ldots (a_n\hat{\chi}_n)(\zeta_n) = \begin{cases} 1 & \text{if } a_1\hat{\chi}_1 = \cdots = a_n\hat{\chi}_n, \\ 0 & \text{otherwise.} \end{cases}$$

In the first case, we set $\hat{\chi} = a_1\hat{\chi}_1 = \cdots = a_n\hat{\chi}_n$ and have $\chi = \check{a}_i\chi_i$ for all i and so $\chi_i^{-1} = \chi^{-1}\check{a}_i$. We deduce the needed result as $\check{a}_i\chi^{-1} \neq \mathbf{1} \iff \chi \neq \check{a}_i$. □

2.3. **Computation of the sums for toric Fermat hypersurfaces.** Just like for $S_{D/\mathbb{F}_q,a}$, we start by computing the fixator.

Proposition 2.5. *We continue to consider a fixed non-trivial additive character φ of $\mathbb{F}_q$ and keep the notations of the beginning of §2. If $[\zeta] = [\zeta_1, \ldots, \zeta_n] \in A$, then*

$$|\mathrm{Fix}_{D^*}(\mathrm{Frob} \circ [\zeta]^{-1})| = \frac{(q-1)^{n-1}}{q} + \frac{1}{q} \sum_{\substack{\chi_i^d=\mathbf{1} \\ \chi_1\cdots\chi_n=\mathbf{1}}} G(\varphi, \chi_1^{-1}) \ldots G(\varphi, \chi_n^{-1})\hat{\chi}_1(\zeta_1) \ldots \hat{\chi}_n(\zeta_n).$$

Proof. The method is the same as in the previous subsection. We compute first the affine fixator and then deduce the projective one thanks to the formula

$$|\mathrm{Fix}^{\mathrm{proj}}_{D^*}(\mathrm{Frob} \circ [\zeta]^{-1})| = \frac{|\mathrm{Fix}^{\mathrm{aff}}_{D^*}(\mathrm{Frob} \circ \zeta^{-1})|}{q-1}. \tag{2.1}$$

As in Proposition 2.3, $\varphi(ax_i^d)$ makes sense when $x_i^q = \zeta_i x_i$. We first obtain sums over non-zero elements:

$$|\mathrm{Fix}^{\mathrm{aff}}_{D^*}(\mathrm{Frob}\circ\zeta^{-1})| = \frac{(q-1)^n}{q} + \frac{1}{q}\sum_{a\in\mathbb{F}_q^*}\Big(\sum_{x_1^{q-1}=\zeta_1}\varphi(ax_1^d)\Big)\dots\Big(\sum_{x_n^{q-1}=\zeta_n}\varphi(ax_n^d)\Big).$$

We now use a Fourier inversion:

$$|\mathrm{Fix}^{\mathrm{aff}}_{D^*}(\mathrm{Frob}\circ\zeta^{-1})| = \frac{(q-1)^n}{q} + \frac{(q-1)}{q}\sum_{\substack{\chi_i^d=\mathbf{1}\\ \chi_1\cdots\chi_n=\mathbf{1}}} G(\varphi,\chi_1^{-1})\dots G(\varphi,\chi_n^{-1})\hat{\chi}_1(\zeta_1)\dots\hat{\chi}_n(\zeta_n),$$

which gives the result after using (2.1). □

Theorem 2.6. *We continue to consider a fixed non-trivial additive character φ of $\mathbb{F}_q$ and keep the notations of the beginning of §2. If $a\in\hat{A}$,*

$$S_{D^*/\mathbb{F}_q,a} = \frac{(q-1)^{n-1}}{q}\,\delta_{a=[0]} + \frac{1}{q}\sum_{\chi^d=\mathbf{1}} G(\varphi,\chi^{-1}\check{a}_1)\dots G(\varphi,\chi^{-1}\check{a}_n).$$

Proof. The principle of the proof is the same as for Theorem 2.4. □

2.4. **Computation of the sums for the complement of toric Fermat hypersurfaces.** Let $S_{D'/\mathbb{F}_q,a} = S_{D/\mathbb{F}_q,a} - S_{D^*/\mathbb{F}_q,a}$ (this is the sum corresponding to the case where at least one of the x_i is zero). We have the following result.

Theorem 2.7. *Fix as before a non-trivial additive character φ of $\mathbb{F}_q$ and keep the notations of the beginning of §2. If $a\in\hat{A}$,*

$$S_{D'/\mathbb{F}_q,a} = \Big(1+q+\cdots+q^{n-2}-\tfrac{(q-1)^{n-1}}{q}\Big)\delta_{a=[0]} - \frac{1}{q}\sum_{\substack{\chi^d=\mathbf{1}\\ \exists i,\ \chi=\check{a}_i}} G(\varphi,\chi^{-1}\check{a}_1)\dots G(\varphi,\chi^{-1}\check{a}_n).$$

Proof. This is an immediate consequence of Theorem 2.4, Theorem 2.6 and of the relation $S_{D'/\mathbb{F}_q,a} = S_{D/\mathbb{F}_q,a} - S_{D^*/\mathbb{F}_q,a}$. □

3. Computation of the sums for Dwork hypersurfaces

Just like for the Fermat hypersurfaces, we may, without any loss of generality, restrict to the computation of $S_{X_\psi/\mathbb{F}_q,a} = S_{X_\psi/\mathbb{F}_q,a,1}$. Let us note that the computations of all this section are valid when n is an integer ≥ 1 satisfying $q\equiv 1 \mod n$.

We go back to the notations and assumptions of the introduction and use the notations $b\mapsto\check{b}$ and $\chi\mapsto\hat{\chi}$ from the beginning of §2 when $d=n$. We denote by X_ψ^* the (projective) toric hypersurface given by the same equation as X_ψ and define the corresponding sum for $a\in\hat{A}$

$$S_{X_\psi^*/\mathbb{F}_q,a} = \frac{1}{|A|}\sum_{[\zeta]\in A} a([\zeta])\,|\mathrm{Fix}_{X_\psi^*}(\mathrm{Frob}\circ[\zeta]^{-1})|.$$

(As before, Frob denotes the Frobenius endomorphism induced by $x \mapsto x^q$.) The method to compute this sum is the same as for the Fermat hypersurface.

From the relation $S_{X_\psi/\mathbb{F}_q,a} - S_{X^*_\psi/\mathbb{F}_q,a} = S_{D/\mathbb{F}_q,a} - S_{D^*/\mathbb{F}_q,a}$ when $d = n$, we will then decude the value of $S_{X_\psi/\mathbb{F}_q,a}$ (this is the reason why we needed to compute the sums for Fermat hypersurfaces).

As in §2, we start by a remark, and notice that Remark 2.1 and Lemma 2.2 both stay valid when $d = n$.

Remark 3.1. Let $(x_i)_{1\leq i\leq n}$ be a finite sequence of elements of $\overline{\mathbb{F}}_q$. If, for each $i \in [\![1, n]\!]$, we can write $x_i^{q-1} = \zeta_i$ with $\zeta_1 \dots \zeta_n = 1$, then $x_1 \dots x_n \in \mathbb{F}_q$. Indeed, $(x_1 \dots x_n)^{q-1} = \zeta_1 \dots \zeta_n = 1$.

3.1. **Computation of the sums for toric Dwork hypersurfaces.** The method is the same as in §2.3 for the toric Fermat hypersurface.

Proposition 3.2. *Fix as before a non-trivial additive character φ of $\mathbb{F}_q$. If $[\zeta] = [\zeta_1, \dots, \zeta_n] \in A$, then*

$$|\mathrm{Fix}_{X^*_\psi}(\mathrm{Frob} \circ [\zeta]^{-1})| = \frac{(q-1)^{n-1}}{q} + \sum_{\substack{\chi_i^n = \mathbf{1} \\ \chi_1\cdots\chi_n = \mathbf{1} \\ \mathrm{mod}\ (\chi,\dots,\chi)}} N_{\chi_1,\dots,\chi_n,\eta}(\psi)\, \hat{\chi}_1(\zeta_1) \dots \hat{\chi}_n(\zeta_n),$$

where

$$N_{\chi_1,\dots,\chi_n,\eta}(\psi) = \frac{1}{q-1} \sum_{\eta \in \widehat{\mathbb{F}_q^*}} \frac{1}{q} G(\varphi, \chi_1^{-1}\eta^{-1}) \dots G(\varphi, \chi_n^{-1}\eta^{-1}) \cdot G(\varphi, \eta^n)\eta(\tfrac{1}{(-n\psi)^n}).$$

The following proof follows closely the corresponding computation of $|X_\psi(\mathbb{F}_q)|$ from [3, §4.2], but we repeat all the arguments in detail.

Proof. Set $f(x) = x_1^n + \dots + x_n^n - n\psi x_1 \dots x_n = 0$ where $\psi \in \mathbb{F}_q^*$ is a parameter. The method is the same as for the Fermat hypersurface (in particular, we first compute affinely and then projectively). Notice that, by Remark 2.1.(1), it makes sense to consider $\varphi(ax_i^n)$ when $x_i^q = \zeta_i x_i$; the same goes for $\varphi(-n\psi a x_1 \dots x_n)$ by Remark 3.1. We write

$$\begin{aligned}|\mathrm{Fix}^{\mathrm{aff}}_{X^*_\psi}(\mathrm{Frob} \circ \zeta^{-1})| &= \frac{1}{q} \sum_{a\in\mathbb{F}_q} \sum_{x_i \in \overline{\mathbb{F}}_q^*,\ x_i^q = \zeta_i x_i} \varphi(af(x)) \\ &= \frac{(q-1)^n}{q} + \frac{1}{q} \sum_{a \in \mathbb{F}_q^*} \sum_{x_i^{q-1} = \zeta_i} \varphi(ax_1^n) \dots \varphi(ax_n^n)\varphi(-n\psi a x_1 \dots x_n).\end{aligned}$$

We now use the Fourier inversion formula for $\varphi|_{\mathbb{F}_q^*}$:

$$\begin{aligned}|\mathrm{Fix}^{\mathrm{aff}}_{X^*_\psi}(\mathrm{Frob} \circ \zeta^{-1})| &= \frac{(q-1)^n}{q} \\ &+ \frac{1}{q}\frac{1}{(q-1)^{n+1}} \sum_{\substack{a \in \mathbb{F}_q^* \\ \eta_1,\dots,\eta_{n+1} \in \widehat{\mathbb{F}_q^*} \\ x_i^{q-1} = \zeta_i}} G(\varphi, \eta_1^{-1}) \dots G(\varphi, \eta_{n+1}^{-1})\eta_1(ax_1^n) \dots \eta_n(ax_n^n) \\ &\qquad \cdot \eta_{n+1}(-n\psi a x_1 \dots x_n).\end{aligned}$$

We extend the characters η_i into characters $\overline{\eta}_i$ of $\overline{\mathbb{F}}_q^*$. The previous sum can be rewritten as

$$\begin{aligned}|\mathrm{Fix}^{\mathrm{aff}}_{X^*_\psi}(\mathrm{Frob}\circ\zeta^{-1})| &= \frac{(q-1)^n}{q} \\ &+ \frac{1}{q}\frac{1}{(q-1)^{n+1}}\sum_{\eta_1,\dots,\eta_{n+1}} G(\varphi,\eta_1^{-1})\dots G(\varphi,\eta_{n+1}^{-1})\Big(\sum_{a\in\mathbb{F}_q^*}(\eta_1\dots\eta_{n+1})(a)\Big) \\ &\times\Big(\sum_{x_1^{q-1}=\zeta_1}(\overline{\eta}_1^n\overline{\eta}_{n+1})(x_1)\Big)\dots\Big(\sum_{x_n^{q-1}=\zeta_n}(\overline{\eta}_n^n\overline{\eta}_{n+1})(x_n)\Big)\eta_{n+1}(-n\psi).\end{aligned}$$

The sum over a is immediate to compute thanks to an orthogonality formula:

$$\sum_{a\in\mathbb{F}_q^*}(\eta_1\dots\eta_{n+1})(a) = \begin{cases} q-1 & \text{if } \eta_1\dots\eta_{n+1}=\mathbf{1}, \\ 0 & \text{if } \eta_1\dots\eta_{n+1}\neq\mathbf{1},\end{cases}$$

and the sums over the x_i can be computed thanks to a change of variable and an orthogonality formula; more precisely, if $\xi_i^{q-1}=\zeta_i$,

$$\sum_{x_i^{q-1}=\zeta_i}(\overline{\eta}_i^n\overline{\eta}_{n+1})(x_i) = \begin{cases}(q-1)(\overline{\eta}_1^n\overline{\eta}_{n+1})(\xi_i) & \text{si } \eta_i^n\eta_{n+1}=\mathbf{1}, \\ 0 & \text{si } \eta_i^n\eta_{n+1}\neq\mathbf{1}.\end{cases}$$

This shows that, in the original sum, we may take away all the terms corresponding to characters η_i which do not satisfy $\eta_1\dots\eta_{n+1}=\mathbf{1}$ and $\eta_i^n\eta_{n+1}=\mathbf{1}$. We consider η such that $\eta^n=\eta_{n+1}^{-1}$, and obtain

$$\begin{cases}\eta_i^n\eta_{n+1}=\mathbf{1} \\ \eta_1\dots\eta_{n+1}=\mathbf{1}\end{cases} \iff \begin{cases}\eta_i=\chi_i\eta \\ \chi_1\dots\chi_n=\mathbf{1}\end{cases} \quad \text{where } \chi_i \text{ satisfies } \chi_i^n=\mathbf{1}.$$

(We also choose the extensions of the characters in a way which is compatible with this system of equations.) The character η is not unique; indeed, if η' and χ_i' are also solution of the system, there exists χ satisfying $\chi^n=\mathbf{1}$ such that $\eta'=\chi^{-1}\eta$ and $\chi_i'=\chi\chi_i$ for all i. This means that if R is a system of representatives of the n-uples $(\chi_1,\dots,\chi_n)$ of characters satisfying both $\chi_i^n=\mathbf{1}$ and $\chi_1\dots\chi_n=\mathbf{1}$ mod the $(\chi,\dots,\chi)$ with $\chi^n=\mathbf{1}$, then the map $(\chi_1,\dots,\chi_n,\eta)\mapsto(\chi_1\eta,\dots,\chi_n\eta,\eta^{-n})$ is a bijection of $R\times\widehat{\mathbb{F}_q^*}$ onto the set of $(n+1)$-uples $(\eta_1,\dots,\eta_{n+1})$ satisfying the previous system. Hence, the sum we began with can be written as

$$\begin{aligned}|\mathrm{Fix}^{\mathrm{aff}}_{X^*_\psi}(\mathrm{Frob}\circ\zeta^{-1})| &= \frac{(q-1)^n}{q} \\ &+ \frac{1}{q}\frac{1}{(q-1)^n}\sum_{\substack{\chi_i^n=\mathbf{1}\\ \chi_1\cdots\chi_n=\mathbf{1}\\ \mathrm{mod}\ (\chi,\dots,\chi)}}\ \sum_{\eta\in\widehat{\mathbb{F}_q^*}}\ \sum_{x_i^{q-1}=\zeta_i} G(\varphi,(\chi_1\eta)^{-1})\dots G(\varphi,(\chi_n\eta)^{-1}) \\ &\qquad\cdot G(\varphi,\eta^n)\chi_1(x_1^n)\dots\chi_n(x_n^n) \\ &\qquad\cdot\eta\big(\tfrac{1}{(-n\psi)^n}\big).\end{aligned}$$

(We have used the fact that $\overline{\chi}_i^n(x_i) = \chi_i(x_i^n)$.) Finally, by Lemma 2.2,

$$|\mathrm{Fix}^{\mathrm{aff}}_{X^*_\psi}(\mathrm{Frob}\circ\zeta^{-1})| = \frac{(q-1)^n}{q} + \frac{1}{q}\sum_{\substack{\chi_i^n=\mathbf{1}\\ \chi_1\cdots\chi_n=\mathbf{1}\\ \bmod (\chi,\ldots,\chi)}}\ \sum_{\eta\in\widehat{\mathbb{F}_q^*}} G(\varphi,\chi_1^{-1}\eta^{-1})\ldots G(\varphi,\chi_n^{-1}\eta^{-1})G(\varphi,\eta^n) \cdot\eta(\tfrac{1}{(-n\psi)^n})\hat{\chi}_1(\zeta_1)\ldots\hat{\chi}_n(\zeta_n).$$

By counting in the projective space (which amounts to a division by $q-1$), we get the announced result. □

Theorem 3.3. *Fix as before a non-trivial additive character φ of $\mathbb{F}_q$. If $a\in\hat{A}$,*

$$S_{X^*_\psi/\mathbb{F}_q,a} = \frac{(q-1)^{n-1}}{q}\delta_{a=[0]} + \frac{1}{q-1}\sum_\eta \frac{1}{q} G(\varphi,\check{a}_1\eta^{-1})\ldots G(\varphi,\check{a}_n\eta^{-1}) \cdot G(\varphi,\eta^n)\eta(\tfrac{1}{(-n\psi)^n}).$$

Proof. The principle is the same as for the Fermat hypersurfaces (Theorem 2.6), namely the use of orthogonality formulas. Let us give a few details on the computation. The sum over the χ_i satisfying $\chi_i^n=\mathbf{1}$ and $\chi_1\ldots\chi_n=\mathbf{1}$ mod the $(\chi,\ldots,\chi)$ is equal to $\frac{1}{n}$ times the sum over the χ_i satisfying $\chi_i^n=\mathbf{1}$ and $\chi_1\ldots\chi_n=\mathbf{1}$. Applying orthogonality formulas, we get

$$\frac{1}{n}\sum_{\chi^d=\mathbf{1}}\sum_{\eta\in\widehat{\mathbb{F}_q^*}} G(\varphi,\check{a}_1\chi^{-1}\eta^{-1})\ldots G(\varphi,\check{a}_n\chi^{-1}\eta^{-1})G(\varphi,\eta^n)\chi(\tfrac{1}{(-n\psi)^n}).$$

The change of variable $\chi\eta\to\eta$ gives the announced formula. □

3.2. **Computation of the sums for Dwork hypersurfaces.** We are now able to compute $S_{X_\psi/\mathbb{F}_q,a}$.

Theorem 3.4. *Fix as before a non-trivial additive character φ of $\mathbb{F}_q$. If $a\in\hat{A}$,*

$$S_{X_\psi/\mathbb{F}_q,a} = (1+q+\cdots+q^{n-2})\delta_{a=[0]} + \frac{1}{q-1}\sum_\eta \frac{1}{q^{\delta_{\forall i,\ \eta\neq\check{a}_i}}} G(\varphi,\check{a}_1\eta^{-1})\ldots G(\varphi,\check{a}_n\eta^{-1})G(\varphi,\eta^n)\eta(\tfrac{1}{(-n\psi)^n}).$$

Proof. Set $S_{X'_\psi/\mathbb{F}_q,a} = S_{X_\psi/\mathbb{F}_q,a} - S_{X^*_\psi/\mathbb{F}_q,a}$. We have $S_{X'_\psi/\mathbb{F}_q,a} = S_{D'/\mathbb{F}_q,a}$ (with $d=n$) because $x_1\ldots x_n=0$ when at least one of the x_i is zero. Hence,

$$S_{X_\psi/\mathbb{F}_q,a} = S_{X^*_\psi/\mathbb{F}_q,a} + S_{D'/\mathbb{F}_q,a},$$

where $S_{X^*_\psi/\mathbb{F}_q,a}$ is given by Theorem 3.3 and $S_{D'/\mathbb{F}_q,a}$ by Theorem 2.7. We can write (notice that, when $\eta^n=\mathbf{1}$, $G(\varphi,\eta^n)=G(\varphi,\mathbf{1})=-1$ and $\eta(\frac{1}{(-n\psi)^n})=1$):

$$S_{X'_\psi/\mathbb{F}_q,a} = \left(1+q+\cdots+q^{n-2}-\frac{(q-1)^{n-1}}{q}\right)\delta_{a=[0]} + \frac{1}{q-1}\sum_{\substack{\eta^n=\mathbf{1}\\ \exists i,\ \eta=\check{a}_i}} \frac{q-1}{q}\left(\prod_{i=1}^n G(\varphi,\chi^{-1}\check{a}_i)\right)G(\varphi,\eta^n)\eta(\tfrac{1}{(-n\psi)^n}).$$

This expression can be included into $S_{X^*_\psi/\mathbb{F}_q,a}$ in a natural way and this gives the announced result. □

4. L FUNCTION OF THE SUMS

Definition 4.1. Let X be a (smooth) variety over $\mathbb{F}_q$, G a finite group of automorphisms acting algebraically on X and ρ a representation of G irreducible over $\overline{\mathbb{Q}}_\ell$. We set

$$S_{X/\mathbb{F}_q,\rho,r} = \frac{1}{|G|}\sum_{g\in G} \operatorname{tr}\rho(g)\,|\mathrm{Fix}(\mathrm{Frob}^r \circ g^{-1})|$$

and build the associated L function

$$L_{X/\mathbb{F}_q,\rho}(t) = \exp\Bigg(\sum_{r=1}^{+\infty} S_{X/\mathbb{F}_q,\rho,r}\,\frac{t^r}{r}\Bigg).$$

Theorem 4.2. *If X is a projective scheme over $\mathbb{F}_q$ which is smooth and of dimension m, then*

$$\sum_{i=0}^{2m}(-1)^i \operatorname{tr}((\mathrm{Frob}^r\circ g^{-1})^*|H^i_{\mathrm{et}}(\overline{X},\mathbb{Q}_\ell)) = |\mathrm{Fix}(\mathrm{Frob}^r\circ g^{-1})|.$$

Proof. See [2, §3 p. 119], which in turn refers to [5, 7]. □

Proposition 4.3. *We keep the preceding notations. Denote by $H^i_{\mathrm{et}}(\overline{X},\overline{\mathbb{Q}}_\ell)^\rho$ the isotypic component of type ρ of $H^i_{\mathrm{et}}(\overline{X},\overline{\mathbb{Q}}_\ell)$ and set $P_{i,\rho}(t) = \det(1 - t\,\mathrm{Frob}^*|$ $H^i_{\mathrm{et}}(\overline{X},\overline{\mathbb{Q}}_\ell)^\rho)$. We have*

$$L_{X/\mathbb{F}_q,\rho}(t)^{\dim\rho} = \prod_{i=0}^{2m} P_{i,\rho}(t)^{(-1)^{i+1}}.$$

Proof. This theorem comes from [6, p. 170–172]; in order to prove it, we just replace the cardinal of the fixator by its value in terms of an alternated sum in the definition of the L function:

$$\begin{aligned}
&L_{X/\mathbb{F}_q,\rho}(t)^{\dim\rho}\\
&\quad= \exp\Bigg(\sum_{i=0}^{2m}(-1)^i\sum_{r=1}^{+\infty}\frac{\dim\rho}{|G|}\sum_{g\in G}\operatorname{tr}\rho(g)\operatorname{tr}\big((\mathrm{Frob}^r\circ g^{-1})^*\big|H^i_{\mathrm{et}}(\overline{X},\overline{\mathbb{Q}}_\ell)\big)\,\frac{t^r}{r}\Bigg)\\
&\quad= \prod_{i=0}^{2m}\exp\Bigg(\sum_{r=1}^{+\infty}\operatorname{tr}\Bigg(\frac{\dim\rho}{|G|}\sum_{g\in G}\operatorname{tr}\rho(g)(g^*)^{-1}\circ(\mathrm{Frob}^*)^r\Bigg|H^i_{\mathrm{et}}(\overline{X},\overline{\mathbb{Q}}_\ell)\Bigg)\,\frac{t^r}{r}\Bigg)^{(-1)^i}\\
&\quad= \prod_{i=0}^{2m}\exp\Bigg(\sum_{r=1}^{+\infty}\operatorname{tr}\big(\pi\circ(\mathrm{Frob}^*)^r\big|H^i_{\mathrm{et}}(\overline{X},\overline{\mathbb{Q}}_\ell)\big)\,\frac{t^r}{r}\Bigg)^{(-1)^i}\\
&\quad= \prod_{i=0}^{2m}\exp\Bigg(\sum_{r=1}^{+\infty}\operatorname{tr}\big((\mathrm{Frob}^*)^r\big|H^i_{\mathrm{et}}(\overline{X},\overline{\mathbb{Q}}_\ell)^\rho\big)\,\frac{t^r}{r}\Bigg)^{(-1)^i}\\
&\quad= \prod_{i=0}^{2m}\left(\frac{1}{P_{i,\rho}(t)}\right)^{(-1)^i},
\end{aligned}$$

where the linear map $\pi = \frac{\dim\rho}{|G|}\sum_{g\in G}[\operatorname{tr}\rho(g)](g^*)^{-1}$ projects $H^i_{\mathrm{et}}(\overline{X},\overline{\mathbb{Q}}_\ell)$ on the isotypic component $H^i_{\mathrm{et}}(\overline{X},\overline{\mathbb{Q}}_\ell)^\rho$. □

Remark 4.4. The decomposition of each $H^i_{\mathrm{et}}(\overline{X}, \overline{\mathbb{Q}}_\ell)$ into isotypic components gives the following decomposition:

$$Z_{X/\mathbb{F}_q}(t) = \prod_{\rho \text{ irred.}/\overline{\mathbb{Q}}_\ell} L_{X/\mathbb{F}_q,\rho}(t)^{\dim \rho}.$$

For the rest of this section, we go back to the situation of the introduction: $X = X_\psi$ (dimension $m = n-2$) and $G = A$. As this group is abelian, we have $\dim \rho = 1$ for all irreducible representation over $\overline{\mathbb{Q}}_\ell$, so the formula of the previous remark is valid without any powers.

Proposition 4.5. *Recall from the introduction that $n \geq 5$ is assumed prime and that $q \equiv 1 \mod n$. Consider $i \in [\![0, n-3]\!]$ and $a \in \hat{A}$. The polynomial $P_{i,a}$ corresponding to X_ψ is equal to 1 except when $a = [0]$ and i is even, in which case $P_{i,[0]} = 1 - q^{i/2}t$.*

Proof. Consider $i \in [\![0, n-3]\!]$ and first assume that i is odd. Because X_ψ is a non-singular projective hypersurface, we have $H^i_{\mathrm{et}}(X_\psi, \overline{\mathbb{Q}}_\ell) = \{0\}$ and so $P_{i,a} = 1$.

Assume now that i is even. The spaces $H^i_{\mathrm{et}}(X_\psi, \overline{\mathbb{Q}}_\ell)$ have dimension 1 and the group A acts trivially on each of them (because the elements of A extend to automorphisms of $\mathbb{P}^{n-1}$; this comes from the fact that $PGL_n(\overline{\mathbb{F}}_q)$ does not admit non-trivial representations of degree 1, see [4, Lemma 2.4]) and thus $P_{i,a} = 1$ if $a \neq [0]$. From $\prod_a P_{i,a} = \det(1 - t\,\mathrm{Frob}^* | H^{n-2}_{\mathrm{et}}(\overline{X}_\psi, \overline{\mathbb{Q}}_\ell)) = 1 - q^{i/2}t$ (Leftschetz polynomials), we deduce that $P_{i,[0]} = 1 - q^{i/2}t$. □

Using Proposition 4.3, we deduce the following.

Corollary 4.6. *If $a \neq [0]$,*

$$L_{X_\psi/\mathbb{F}_q,a}(t) = P_{n-2,a}(t) = \det(1 - t\,\mathrm{Frob}^* | H^{n-2}_{\mathrm{et}}(\overline{X}_\psi, \overline{\mathbb{Q}}_\ell)^a).$$

In particular, $L_{X_\psi/\mathbb{F}_q,a}(t)$ is a polynomial.

Remark 4.7. When $a = [0]$, the behaviour is different: the L function $L_{X_\psi/\mathbb{F}_q,[0]}(t)$ is the zeta function of the mirror variety of X_ψ, see [8, Lemma 7.2, p. 174] and is thus a rational function which is not a polynomial.

5. Comparison of the two factorizations

Notations. We begin by introducing the following notations for $a \in \hat{A}$:

- $\langle a \rangle$ the class of $a \in A$ mod $\mathfrak{S}_n$;
- $\overline{a}$ the class of $a \in A$ mod $(\mathbb{Z}/n\mathbb{Z})^\times$;
- $\underline{a}$ the class of $a \in A$ mod the simultaneous actions of $\mathfrak{S}_n$ and $(\mathbb{Z}/n\mathbb{Z})^\times$.

Remark 5.1. Note that m_a and γ_a from the introduction only depend on $\underline{a}$ and that k_a and $S_{\overline{a}}$ only depend on $\overline{a}$.

5.1. **Relation between L functions and the explicit factorization.** Before defining what R_a is, let us recall a few results from [3].

As before, n denotes a prime number ≥ 5 satisfying $q \equiv 1 \mod n$. If $a \in \hat{A}$ and χ is a fixed character of order n of $\mathbb{F}_q^*$, set

$$N_{\underline{a}} = |\mathrm{Im}\, k_a| \sum_{\langle a' \rangle \in \underline{a}} \frac{1}{q-1} \sum_{\eta \in \widehat{\mathbb{F}_q^*}} \left(\prod_{i=1}^{n} q^{-\delta_{\forall i,\ \chi^{-a_i} \neq \eta}} G(\varphi, \chi^{-a_i}\eta^{-1}) \right) \cdot G(\varphi, \eta^n)\eta\left(\tfrac{1}{(-n\psi)^n}\right).$$

(Compare this formula with Theorem 3.4 when $\check{a}_i = \chi^{-a_i}$.) With this notation, we have [3, §4.2]

$$|X_\psi(\mathbb{F}_q)| = 1 + q + \cdots + q^{n-2} + \sum_{\underline{a}} \frac{\gamma_a}{|\mathrm{Im}\, k_a|} N_{\underline{a}},$$

where $N_{\underline{(0,1,2,\ldots,n-1)}} = 0$ because $\psi^n \neq 1$ (see [3, §4.4]). Moreover, by [3, §5.3 and §4.2], there exists an affine hypersurface H_a of hypergeometric type and odd dimension $\leq n-4$ such that

$$N_{\underline{a}} = q^{\frac{n-2-\dim H_a}{2}} (|H_a(\mathbb{F}_q)| - q^{\dim H_a}).$$

The hypergeometric hypersurfaces H_a have explicit equations of the form

$$y^n = x_1^{\alpha_1} \ldots x_d^{\alpha_d} (1-x_1)^{\beta_1} \ldots (1 - x_k - 1)^{\beta_{k-1}} \\ \cdot (1 - x_k - \cdots - x_d)^{\beta_k} (1 - \tfrac{1}{\psi^n} x_1 \ldots x_k)^{\gamma},$$

where the integers α_i, β_i and γ depend on a.

Definition 5.2. $R_a(t) = \exp\left(\displaystyle\sum_{r=1}^{+\infty} N_{\underline{a}}(t) \frac{t^r}{r}\right)$.

Theorem 5.3. *Assume that $a \neq [0]$. Using the notations from §4, we have*

$$N_{\underline{a}} = |\mathrm{Im}\, k_a| \sum_{\langle a' \rangle \in \underline{a}} S_{X_\psi/\mathbb{F}_q, a'}, \quad \textit{hence} \quad R_a(t) = \left(\prod_{\langle a' \rangle \in \underline{a}} L_{X_\psi/\mathbb{F}_q, a'}(t) \right)^{|\mathrm{Im}\, k_a|}.$$

In particular, R_a is a polynomial.

Proof. This is just a reformulation of Theorem 3.4 using the notations we have just introduced. □

5.2. **Relation between L functions and the cohomological factorization.** We use the notations of the introduction concerning Q_a and W_a.

Theorem 5.4. *Assume that $a \neq [0, \ldots, 0]$. We have*

$$Q_a = \prod_{\langle a' \rangle \in \underline{a}} L_{X_\psi/\mathbb{F}_q, a'}(t).$$

Proof. Denote by $\overline{H}_a$ the isotypic component of type a of the $\overline{\mathbb{Q}}_\ell[A]$-module $H^{n-2}_{\mathrm{et}}(\overline{X}_\psi, \overline{\mathbb{Q}}_\ell)$ and H_{W_a} the isotypic component of type W_a of the $\mathbb{Q}[A \rtimes \mathfrak{S}_n]$-module $H^{n-2}_{\mathrm{et}}(\overline{X}_\psi, \mathbb{Q}_\ell)$. By Theorem 5.16 of [4], we have

$$W_a \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \simeq \bigoplus_{a' \in \underline{a}} a' \quad \text{hence} \quad H_{W_a} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \simeq \bigoplus_{a' \in \underline{a}} \overline{H}_{a'}.$$

(The first isomorphism is an isomorphism of $\overline{\mathbb{Q}}_\ell[A]$-modules whereas the second one is an isomorphism of $\overline{\mathbb{Q}}_\ell[A \rtimes \mathfrak{S}_n]$-modules; indeed, each $\overline{H}_{a'}$ is only a $\overline{\mathbb{Q}}_\ell[A]$-module but their sum becomes a $\overline{\mathbb{Q}}_\ell[A \rtimes \mathfrak{S}_n]$-module.) Therefore, as $L_{X_\psi/\mathbb{F}_q, a}(t) = \det(1 - t\,\mathrm{Frob}^* | \overline{H}_a)$ (Corollary 4.6) and $Q_a(t)^{\gamma_a} = \det(1 - t\,\mathrm{Frob}^* | H_{W_a}) = \det(1 - t\,\mathrm{Frob}^* \otimes \mathrm{Id} | H_{W_a} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$,

$$Q_a(t)^{\gamma_a} = \prod_{a' \in \underline{a}} L_{X_\psi/\mathbb{F}_q, a'}(t) = \left(\prod_{\langle a' \rangle \in \underline{a}} L_{X_\psi/\mathbb{F}_q, a'}(t) \right)^{\gamma_a}.$$

(We have used the fact that $S_{X_\psi/\mathbb{F}_q,a'} = S_{X_\psi/\mathbb{F}_q,a}$ if a' is a permutation of a.) Because $Q_a(t)$ and $\prod_{\langle a'\rangle\in\underline{a}} L_{X_\psi/\mathbb{F}_q,a'}(t)$ both belong to $1+t\mathbb{Q}[t]$, we deduce the equality without the power γ_a. □

Corollary 5.5. *We have*

$$R_a(t) = Q_a(t)^{|\operatorname{Im} k_a|} \quad \textit{and so} \quad \deg R_a = (n-1)m_a.$$

Remark 5.6. Before we continue, let us make a few remarks and recall some results from [4].

(1) As n is prime, an element $a \in \hat{A}$ which is $\neq [0]$ has order n in the group $\hat{A}$. Considered as a character, a thus takes its values in a cyclotomic field $\mathbb{K}_a$ of degree $n-1$ over $\mathbb{Q}$, which we will consider as a subfield of $\overline{\mathbb{Q}}_\ell$, following the identifications made in the introduction (see [4, §5.2] for an intrinsic construction of this field). With this convention, $S_{X_\psi,a,r} \in \mathbb{K}_a$ and thus $L_{X_\psi,a}(t) \in \mathbb{K}_a[t]$.

(2) Denote by D_a the subfield of $\mathbb{K}_a$ fixed by the automorphisms, indexed by $v \in \operatorname{Im} k_a$, which send an n-th root of unity onto its v-th power. This (commutative) field is (isomorphic to) the endomorphism ring of W_a (see [4, Theorem 5.14]). As $v{\cdot}a$ is a permutation of a if $v \in \operatorname{Im} k_a$, we deduce that $S_{X_\psi,va,r} = S_{X_\psi,a,r}$ for all $v \in \operatorname{Im} k_a$, hence, $S_{X_\psi,a,r} \in D_a$ and $L_{X_\psi,a}(t) \in D_a[t]$.

(3) If $\langle a'\rangle \in \underline{a}$ and $v \in (\mathbb{Z}/n\mathbb{Z})^\times$, the formula

$$S_{X_\psi/\mathbb{F}_q,va,r} = \frac{1}{|A|}\sum_{[\zeta]\in A} a([\zeta])^v \,|\mathrm{Fix}(\mathrm{Frob}^r \circ [\zeta]^{-1})|$$

shows that the sums $S_{X_\psi,a,r}$ are conjugates and hence the polynomials $L_{X_\psi,a}(t)$ are also conjugates. This shows that

$$Q_a = N_{\mathbb{K}_a/D_a}(L_{X_\psi/\mathbb{F}_q,a}(t)).$$

(4) Recall from [4, Proposition 6.6] that $Q_a = N_{\mathbb{K}_a/D_a}(P_a)$ with $P_a = P_{a,1}$ the characteristic polynomial of the Frobenius acting by $v \mapsto \mathrm{Frob}^* \circ v$ on the $D_a \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module $V_a = V_{a,1} = \operatorname{Hom}_{\mathbb{Q}[A\rtimes\mathfrak{S}_n]}(W_a, H^{n-2}_{\mathrm{et}}(\overline{X}_\psi,\mathbb{Q}_\ell))$, and that $H_{W_a} \simeq W_a \otimes_{D_a} V_a$ (see [4, §6.1]). The two polynomials $P_a(t)$ and $L_{X_\psi/\mathbb{F}_q,a}(t)$ both belong to $D_a[t]$ and have the same degree. They are in fact equal, as the next proposition shows.

Proposition 5.7. *With the notations of Remark 5.6, there exists a suitable embedding of* $\operatorname{End}_{\mathbb{Q}[A\rtimes\mathfrak{S}_n]}(W_a)$ *onto* $D_a \subset \overline{\mathbb{Q}}_\ell$ *such that*

$$P_a = L_{X_\psi/\mathbb{F}_q,a}(t)\,.$$

Proof. Denote by Frob_{W_a} the Frobenius acting on $H_{W_a} = H^{n-2}_{\mathrm{et}}(\overline{X}_\psi,\mathbb{Q}_\ell)^{W_a}$ and considered as a D_a-linear map, $\overline{\mathrm{Frob}}_{W_a}$ the Frobenius acting on $H_{W_a}\otimes_{\mathbb{Q}_\ell}\overline{\mathbb{Q}}_\ell$ and considered as a $\overline{\mathbb{Q}}_\ell$-linear map and $\overline{\mathrm{Frob}}_a$ the Frobenius acting on $\overline{H}_a = H^{n-2}_{\mathrm{et}}(\overline{X}_\psi,\overline{\mathbb{Q}}_\ell)^a$ and considered as a $\overline{\mathbb{Q}}_\ell$-linear map. We are going to show that there exists an embedding β of $\operatorname{End}_{\mathbb{Q}[A\rtimes\mathfrak{S}_n]}(W_a)$ into $\overline{\mathbb{Q}}_\ell$ such that, if $(\delta_{i,j})_{1\le i,j\le m_a}$ is the matrix of Frob_{W_a}, then $(\beta(\delta_{i,j}))_{1\le i,j\le m_a}$ is that of $\overline{\mathrm{Frob}}_a$, which will show the announced result.

We build the embedding β as follows. Let δ_a be a primitive element of the extension $\operatorname{End}_{\mathbb{Q}[A\rtimes\mathfrak{S}_n]}(W_a)/\mathbb{Q}$; after extension of the scalars to $\overline{\mathbb{Q}}_\ell$, the map $\delta_a \otimes$

Id becomes diagonal in every basis adapted to the decomposition $W_a \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell = \bigoplus_{a' \in \underline{a}} a'$; we denote by $\lambda_{a'}$ the eigenvalue corresponding to a' and consider β the embedding of $\mathrm{End}_{\mathbb{Q}[A \rtimes \mathfrak{S}_n]}(W_a)$ into $\overline{\mathbb{Q}}_\ell$ given by $\beta(\delta_a) = \lambda_a$.

The matrix of $\overline{\mathrm{Frob}}_{W_a}$ in the previous basis is $(\delta_{i,j} \otimes \mathrm{Id})_{1 \le i,j \le m_a}$; we write $\delta_{i,j} = \sum_{k=0}^{r-1} \alpha_{i,j,k} \delta_a^k$ with $r = \dim_{\mathbb{Q}} D_a$ and $\alpha_{i,j,k} \in \mathbb{Q}$ and consider again the decomposition $W_a \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell = \bigoplus_{a' \in \underline{a}} a'$. The $\overline{\mathbb{Q}}_\ell$-linear map $\delta_{i,j} \otimes \mathrm{Id}$ induced on the factor which is isomorphic to a' acts by multiplication by $\sum_{k=0}^{r-1} \alpha_{i,j,k} \lambda_{a'}^k$, expression which is equal to $\beta(\delta_{i,j})$ when $a' = a$. The matrix of $\overline{\mathrm{Frob}}_a$ is thus $(\beta(\delta_{i,j}))_{1 \le i,j \le m_a}$, which ends the proof. $\square$

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